AGNAGN seminar
chapter 3. Thermal Equilibrium

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Preface

- Refer to original AGNAGN textbook when discussing tables or figures.
- I've tried to fill in every gap I found in original textbook.
- Therefore, this slide consists of 60% original contents and 40% my own interpretation, derivation , or explanation.
- Since 40% my own interpretation may include some incorrect points, I would appreciate it if you tell me any mistake or suspicious point.

3.1 introduction

- heating : photoionazation (G)
- cooling : recombination(L), cooling by line emission $\gg$ free-free emission

In the equilibrium,

$$G - L = (\text{line}) + (\text{free-free}) \tag{1}$$

3.2 Input by Photoionization

When an ionizing photon ($h\nu$) is absorbed, an electron is produced with kinetic energy $\frac{1}{2}mu^2$. In this process, energy conservation is

$$h\nu = h\nu_0 + \frac{1}{2}mu^2, \quad h\nu_0 = 13.6\text{eV}(\text{ionizing energy}) \tag{2}$$

By integrating this kinetic energy, **total energy input** by photoionization is

$$G(\text{H}) = \int_{\text{ionizing}} \#(\text{photoelectron}) \times \frac{1}{2}mu^2 \tag{3}$$

$$= \int_{\nu_0}^{\infty} n(\text{H}^0)\#(\text{photon})h(\nu - \nu_0)a_\nu(\text{H}^0)d\nu \tag{4}$$

where, $n(\text{H}^0)$: number density of H^0, $a_\nu(\text{H}^0)$: cross section.
Assume that the H^0 gas is under the radiation field J_ν(intensity),

$$\#(\text{photon}) = \frac{4\pi J_\nu}{h\nu} \tag{5}$$

3.2 Definition of T_i

Therefore, the total energy input (per unit volume, unit time) is

Energy input by photoionization

$$G(\mathrm{H}) = \int_{\nu_0}^{\infty} n(\mathrm{H}^0) \frac{4\pi J_\nu}{h\nu} h(\nu - \nu_0) a_\nu(\mathrm{H}^0) d\nu \qquad (6)$$

Assume **ionization equilibrium** (number of H^0 is balanced)

$$n(\mathrm{H}^0) \int_{\nu_0}^{\infty} \frac{4\pi J_\nu}{h\nu} a_\nu d\nu = n_p n_e \alpha_A(\mathrm{H}^0, T) \qquad (7)$$

the total energy input is written as follows

$$G(\mathrm{H}) = n_e n_p \alpha_A(\mathrm{H}^0, T) \frac{\int_{\nu_0}^{\infty} \frac{4\pi J_\nu}{h\nu} h(\nu - \nu_0) a_\nu(\mathrm{H}^0) d\nu}{\int_{\nu_0}^{\infty} \frac{4\pi J_\nu}{h\nu} a_\nu d\nu} \qquad (8)$$

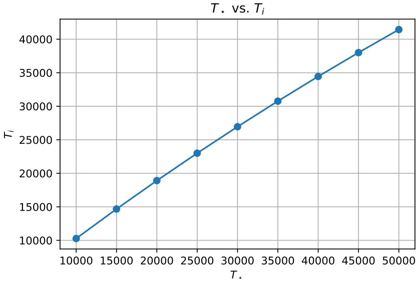
$$=: n_e n_p \alpha_A(\mathrm{H}^0, T) \frac{3}{2} k T_i \qquad (9)$$

3.2 Interpretation of T_i

RHS of eq(9) is the definition of T_i, which reflects the "feeling" that eq(8) represents the "average" of $\frac{1}{2}mu^2$ and it corresponds to internal energy.

simple example

> Consider a simple case; $J_\nu = B_\nu(T_\star), kT_\star < h\nu_0, a_\nu \propto \nu^{-3}$.
> One can numerically calculate the integrations, and get $T_i \simeq T_\star$.
> So, this case is one evidence to regard T_i as the temperature of photoelectrons.



3.2 Hardening(T_i depends on the distance from the star)

We can derive the spatial distribution of T_i.
According to radiative transfer, $J_\nu(r) = J_\nu(0)e^{-\tau_\nu} = J_\nu(0)e^{-na_\nu r}$.
Therefore,

$$\frac{3}{2}kT_i(r) = \frac{\int_{\nu_0}^{\infty} \frac{4\pi J_\nu(0)}{h\nu} e^{-na_\nu r} h(\nu - \nu_0) a_\nu d\nu}{\int_{\nu_0}^{\infty} \frac{4\pi J_\nu(0)}{h\nu} e^{-na_\nu r} a_\nu d\nu} \tag{10}$$

Then we use an inequality (next slide) and we get

$$\frac{3}{2}kT_i(r + dr) = \frac{\int_{\nu_0}^{\infty} \frac{4\pi J_\nu(r)}{h\nu} e^{-na_\nu dr} h(\nu - \nu_0) a_\nu d\nu}{\int_{\nu_0}^{\infty} \frac{4\pi J_\nu(r)}{h\nu} e^{-na_\nu dr} a_\nu d\nu} > \frac{3}{2}kT_i(r) \tag{11}$$

Note that a_ν is known to be decreasing (by Fermi's golden rule), so $e^{-na_\nu dr}$ is increasing.
So, **at larger distance from the star, mean energy of photoelectrons $\frac{3}{2}kT_i(r)$ becomes larger.**

Lemma(redistribution of probability)

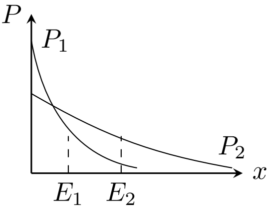
X:random variable

$$P_1(x) = f(x)/Z_1, \quad Z_1 = \int_{\mathbb{R}} f(x)dx \tag{12}$$

$$P_2(x) = f(x)g(x)/Z_2, \quad Z_2 = \int_{\mathbb{R}} f(x)g(x)dx \tag{13}$$

If $g(x)$ is increasing, then

$$\mathbb{E}_{P_1}[X] < \mathbb{E}_{P_2}[X] \tag{14}$$



3.2 Hardening(T_i depends on the distance from the star)

Qualitative explanation of this hardening

- a_ν: smaller frequency is effectively absorbed.
- high energy (high frequency) photons penetrate longer distance.
- Therefore, mean energy (T_i) at larger r becomes higher.

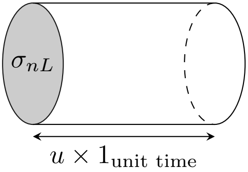
3.3 Energy Loss by Recombination

When an electron falls into a quantum state (n, L), its kinetic energy $\frac{1}{2}mu^2$ is lost. By taking the average of $\frac{1}{2}mu^2$ in velocity space,

$$\int_0^\infty \frac{1}{2}mu^2 \sigma_{nL}(\mathrm{H}^0, T) u f(u) du \tag{15}$$

where, σ_{nL}: cross section , $f(u)$: probability density (often Maxwell-Boltzmann distribution is used).
Note that $\sigma_{nL} \cdot u$ means the volume of cylinder, and

$$\sigma_{nL} \cdot u \times f(u)du = \#(\text{recombine with velocity } u) \tag{16}$$



3.3 Energy Loss by Recombination

Suming up mean energies (15) over quantum states (n, L), total energy loss by recombination is

Energy loss by recombination

$$L_R(\mathrm{H}) = n_e n_p kT \beta_A(\mathrm{H}^0, T) \tag{17}$$

where

$$\beta_A(\mathrm{H}^0, T) = \sum_{n=1}^{\infty} \beta_n(\mathrm{H}^0, T) = \sum_{n=1}^{\infty} \sum_{L=0}^{n-1} \beta_{nL}(\mathrm{H}^0, T) \tag{18}$$

with

$$\beta_{nL}(\mathrm{H}^0, T) = \frac{1}{kT} \int_0^{\infty} u \sigma_{nL}(\mathrm{H}^0, T) \frac{1}{2} m u^2 f(u) du \tag{19}$$

Note that the denominator kT of eq(19) is just for making it dimensionless.

3.3 Inequality between T, T_i

Since $\sigma_{nL} \propto u^{-2}$, electrons with lower kinetic energy are likely to be captured. Therefore,

$$\text{(mean captured energy)} < \text{(mean electron energy)} \quad \rightarrow \quad \beta_A kT < \frac{3}{2}kT \tag{20}$$

In a pure H nebula that had no radiation losses, the thermal equilibrium is

$$G(\mathrm{H}) = L_R(\mathrm{H}) \tag{21}$$

$$n_e n_p \alpha_A(\mathrm{H}^0, T)\frac{3}{2}kT_i = n_e n_p \beta_A(\mathrm{H}^0, T)kT \tag{22}$$

$$\frac{3}{2}kT_i = \beta_A(\mathrm{H}^0, T)kT < \frac{3}{2}kT \tag{23}$$

Therefore, we get $T_i < T$.

3.3 Decomposition of G, L_R

In general, the radiation field J_ν consists of stellar radiation and diffuse radiation ($J_\nu = J_{\nu s} + J_{\nu d}$). So, we can rewrite $G(\mathrm{H})$ as follows

$$G(\mathrm{H}) = \int_{\nu_0}^{\infty} n(\mathrm{H}^0) \frac{4\pi(J_{\nu s} + J_{\nu d})}{h\nu} h(\nu - \nu_0) a_\nu(\mathrm{H}^0) d\nu =: G_s + G_d \tag{24}$$

Also, we can decompose $L_R(\mathrm{H})$ as follows,

$$L_R(\mathrm{H}) = n_e n_p kT(\beta_B + \beta_1), \quad \text{where} \quad \beta_B = \sum_{n=2}^{\infty} \beta_n \tag{25}$$

Note that $\beta_A = \beta_B + \beta_1$.

3.3 On-the-spot approximation

on-the-spot approximation

Photon emitted during a recombination to $n = 1$ level is instantly absorbed by a nearby spot. Energy gain by $J_{\nu d}$ and energy loss by β_1 can simply be omitted.

Using this approximation,

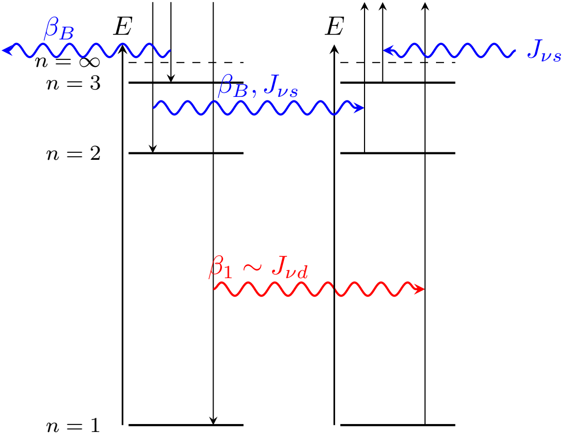
$$G_{\mathrm{OTS}}(\mathrm{H}) = G_s = \int_{\nu_0}^{\infty} n(\mathrm{H}^0) \frac{4\pi J_{\nu s}}{h\nu} h(\nu - \nu_0) a_\nu(\mathrm{H}^0) d\nu \tag{26}$$

$$= n_e n_p \alpha_B(\mathrm{H}^0, T) \frac{\int_{\nu_0}^{\infty} \frac{4\pi J_{\nu s}}{h\nu} h(\nu - \nu_0) a_\nu(\mathrm{H}^0) d\nu}{\int_{\nu_0}^{\infty} \frac{4\pi J_{\nu s}}{h\nu} a_\nu(\mathrm{H}^0) d\nu} \tag{27}$$

and

$$L_{\mathrm{OTS}}(\mathrm{H}) = n_e n_p kT \beta_B(\mathrm{H}^0, T) \tag{28}$$

3.3 Rough illustration of on-the-spot approximation



$E_n = -13.6\text{eV}/n^2$

3.3 Effect of He

The generalization to include He.

$$G = G(\mathrm{H}) + G(\mathrm{He}), \quad L_R = L_R(\mathrm{H}) + L_R(\mathrm{H_e}) \tag{29}$$

where

$$G(\mathrm{He}) = n_e n(\mathrm{He}^+)\alpha_A(\mathrm{He}^0, T)\frac{\int_{\nu_2}^{\infty} \frac{4\pi J_\nu}{h\nu} h(\nu - \nu_2) a_\nu(\mathrm{He}^0) d\nu}{\int_{\nu_2}^{\infty} \frac{4\pi J_\nu}{h\nu} a_\nu(\mathrm{He}^0) d\nu} \tag{30}$$

and

$$L_R(\mathrm{H}) = n_e n(\mathrm{He}^+) kT \beta_A(\mathrm{He}^0, T) \tag{31}$$

Contributions of other atoms are much less since both G and L_R are proportional to density n.

3.4 Energy Loss by Free-Free Radiation

The rate of cooling by free-free radiation by charge Z is

$$L_{\mathrm{FF}}(Z) = 4\pi j_{\mathrm{ff}} \tag{32}$$

$$= \frac{2^5 \pi e^6 Z^2}{3^{3/2} hmc^3} \left(\frac{2\pi kT}{m} \right)^{1/2} g_{\mathrm{ff}} n_e n_+ \tag{33}$$

$$= 1.42 \times 10^{-27} Z^2 T^{1/2} g_{\mathrm{ff}} n_e n_+ \tag{34}$$

according to electromagnetism[1].
Here, g_{ff} is called mean Gaunt factor, $1.0 < g_{\mathrm{ff}} < 1.5$.

[1] cf) Radiative Process in Astrophysics by Rybicki & Lightman

3.5 Energy Loss by Collisionally Excited Line Radiation

Ions with energy level gaps $\sim kT$ such as O^+, O^{++}, N^+ are important sources of cooling.
Here, we consider electrons collide with ions and excite the level 1 to 2.
(**energy gap** $\chi = h\nu_{12}$)

definition of Ω

Excitation cross section of this process is

$$\sigma_{12}(u) =: \begin{cases} \dfrac{\pi\hbar^2}{m^2u^2}\dfrac{\Omega(1,2)}{\omega_1} & \text{for} \quad \dfrac{1}{2}mu^2 > \chi \\ 0 & \text{for} \quad \dfrac{1}{2}mu^2 < \chi \end{cases} \tag{35}$$

where ω_1 is the statistical weight of the lower level.
Here **we define collision strength** $\Omega(1,2)$.

The fact that $\sigma \propto u^{-2}$ reflects **Coulomb focusing effect** (quantum scattering theory).

3.5 Assumption to derive symmetry of Ω

Assume **detailed balancing** between excitation and deexcitation.

$$n_e n_1 u_1 \sigma_{12}(u_1) f(u_1) du_1 = n_e n_2 u_2 \sigma_{21}(u_2) f(u_2) du_2 \tag{36}$$

Note again that σu means the volume of the cylinder through which an electron goes.
Also, u_1 and u_2 satisfy energy conservation

$$\frac{1}{2} m u_1^2 = \frac{1}{2} m u_2^2 + \chi \quad (\rightarrow u_1 du_1 = u_2 du_2) \tag{37}$$

Then, we set another assumption : Boltzmann distribution

$$\frac{n_2}{n_1} = \frac{\omega_2}{\omega_1} \exp(-\chi/kT) \tag{38}$$

Combining eq(36), (37),(38) and substituting $f(u) \propto u^2 \exp\left(-\frac{1}{2} m u^2/kT\right)$,

3.5 Symmetry of Ω

$$\sigma_{12}(u_1)u_1^2 \exp\left(-\frac{1}{2}mu_1^2/kT\right) = \frac{\omega_2}{\omega_1}\exp(-\chi/kT)\sigma_{21}(u_2)u_2^2 \exp\left(-\frac{1}{2}mu_2^2/kT\right) \tag{39}$$

$$\omega_1 u_1^2 \sigma_{12}(u_1) = \omega_2 u_2^2 \sigma_{21}(u_2) \tag{40}$$

Thus, the deexcitation cross section is derived as

$$\sigma_{21}(u_2) = \frac{\omega_1}{\omega_2}\frac{u_1^2}{u_2^2}\sigma_{12(u_1)} \tag{41}$$

$$= \frac{\pi\hbar^2}{m^2 u_2^2}\frac{\Omega(1,2)}{\omega_2} \tag{42}$$

Remembering the definition of Ω, we get

symmetry of Ω

$$\Omega(1,2) = \Omega(2,1) \quad (\Omega \text{ is symmetric}) \tag{43}$$

The total collisional **deexcitation rate** is

$$n_e n_2 q_{21} = n_e n_2 \int_0^\infty u\sigma_{21}(u) f(u) du \tag{44}$$

$$= n_e n_2 \left(\frac{2\pi}{kT}\right)^{1/2} \frac{\hbar^2}{m^{3/2}} \frac{\Upsilon(1,2)}{\omega_2} \tag{45}$$

where, Υ is the velocity-space-averaged collision strength

$$\Upsilon(1,2) = \int_0^\infty \Omega(1,2;E) \cdot \exp(-E/kT) d\left(\frac{E}{kT}\right), \quad \text{with} \quad E = \frac{1}{2} m u_2^2 \tag{46}$$

Also, we can calculate q_{12} with ease as

$$q_{12} = \frac{\omega_2}{\omega_1} q_{21} \exp(-\chi/kT) \tag{47}$$

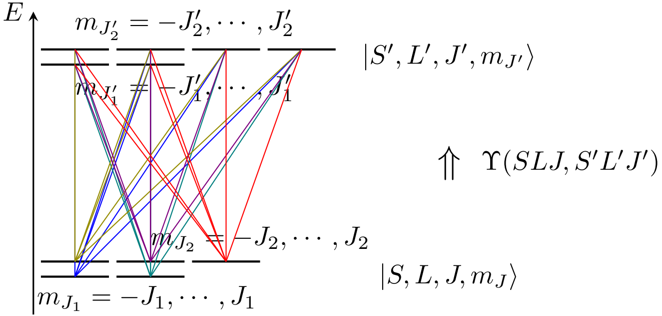
Υ must be calculated with **quantum mechanics**.

3.5 Relation fine-structure-ϒs satisfy

Now we consider the transition from (S, L, J) to (S', L', J') (fine structure).
;J is coupled angular momentum (satisfies "triangle rule")

$$\Upsilon(SLJ, S'L'J') = (\text{collision strength from } (S, L, J) \text{ to } (S', L', J')) \tag{48}$$

$$\Upsilon(SL, S'L') = \sum_{J,J'} \Upsilon(SLJ, S'L'J') \tag{49}$$



3.5 Relation fine-structure-Υs satisfy

In general, the summation of $\Upsilon(SL, S'L')$ is complex.
If

- $S = 0$, then $|L - S| \leq J \leq L + S \rightarrow J = L$.
- $L = 0$, then $|L - S| \leq J \leq L + S \rightarrow J = S$.

In those cases, summation is simplified $\sum_{J,J'} = \sum_{J'}$.

Assume that all collision strength from $|S, L, J, m_J\rangle$ to $|S', L', J', m_{J'}\rangle$ are the same, $\Upsilon(SLJ, S'L'J') \propto 2J' + 1$(degeneracy). Therefore, we get

relation between fine-structure-Υs

$$\Upsilon(SLJ, S'L'J') = \frac{2J'+1}{\sum_{J'=|L-S|}^{L+S}(2J'+1)}\Upsilon(SL, S'L') \quad (50)$$

$$= \frac{2J'+1}{(2S'+1)(2L'+1)}\Upsilon(SL, S'L') \quad (51)$$

example

Consider $^1S \to {}^3P$ (Notation; $^{2S+1}L_J$). The relations are

$$\Upsilon(^1S, {}^3P_0) = \frac{1}{9}\Upsilon(^1S, {}^3P) \tag{52}$$

$$\Upsilon(^1S, {}^3P_1) = \frac{3}{9}\Upsilon(^1S, {}^3P) \tag{53}$$

$$\Upsilon(^1S, {}^3P_2) = \frac{5}{9}\Upsilon(^1S, {}^3P) \tag{54}$$

These relations suggest that the rate of the excitation is nearly independent of the distribution of ions among $^3P_0$, $^3P_1$, and $^3P_2$.

Usage of this relation is rather important in terms of technical reasons.

3.5 Derivation of cooling rate

We can write down **the detailed balancing** between excitation, deexcitation, and spontaneous emission

$$n_e n_1 q_{12} h\nu_{12} = n_e n_1 q_{21} h\nu_{12} + n_2 A_{21} h\nu_{12} \tag{55}$$

$$n_e n_1 q_{12} = n_e n_1 q_{21} + n_2 A_{21} \tag{56}$$

Thus,

$$\frac{n_2}{n_1} = \frac{n_e q_{12}}{A_{21}} \left[1 + \frac{n_e q_{21}}{A_{21}}\right]^{-1} \tag{57}$$

And therefore

cooling rate

$$L_C = n_2 A_{21} h\nu_{12} = n_e n_1 q_{12} h\nu_{12} \left[1 + \frac{n_e q_{21}}{A_{21}}\right]^{-1} \tag{58}$$

3.5 Property of the cooling rate L_C

- $n_e \to 0$

$$L_C = n_e n_1 q_{12} h\nu_{12} \left[1 + \frac{n_e q_{21}}{A_{21}} \right]^{-1} \to n_e n_1 q_{12} h\nu_{12} \tag{59}$$

Explaining qualitatively, all collisionally excited electrons are immediately deexcited by spontaneous emission.

- $n_e \to \infty$

$$L_C \to n_e n_1 q_{12} h\nu_{12} \frac{A_{21}}{n_e q_{21}} = \frac{q_{12}}{q_{21}} n_1 A_{21} h\nu_{21} \tag{60}$$

$$= \frac{\omega_2}{\omega_1} e^{-\chi/kT} n_1 A_{21} h\nu_{21} \quad \text{(using eq(47))} \tag{61}$$

This result is natural considering Boltzmann distribution $\frac{n_2}{n_1} = \frac{\omega_2}{\omega_1} e^{-\chi/kT}$.

3.5 More complex energy levels

Some ions (O^{++}, N^{+} , etc) have more complex energy levels like ^3P. In such cases, the equilibrium equations are

$$\sum_{j\neq i} n_j n_e q_{ji} + \sum_{j>i} n_j A_{ji} = \sum_{j\neq i} n_i n_e q_{ij} + \sum_{j<i} n_i A_{ij} \quad \text{for } \forall i \tag{62}$$

And, cooling rate

$$L_C = \sum_i L_C^{(i)} = \sum_i n_i \sum_{j<i} A_{ij} h\nu_{ij} \tag{63}$$

- $n_e \to 0$, L_C becomes a sum of terms like eq(59).
- $n_e q_{ij} > \sum_{k<i} A_{ik}$, collisional deexcitation is not negligible.

3.5 Critical density

critical density

$$n_c(i) = \sum_{j<i} A_{ij} \; / \sum_{j \neq i} q_{ij} \tag{64}$$

- $n_e < n_c(i)$, collisional deexcitation of level i is negligible.
- $n_e > n_c(i)$, it is not negligible.

3.5 Energy Loss by Collisionally Excited Line Radiation of H

- H^+ has no bound level, no emission.
- H^0 is not so abundant, but may affect the radiative cooling in nebula.
- Important excitation from the ground 1^2S
  - 2^2P^0. Lyα with $h\nu = 10.2$eV
  - 2^2S. 2γ decay with $h\nu' + h\nu'' = 10.2$eV
- Cross section does not vary as u^{-2}. It has resonance and peak structure .
- Υs vary fairly slowly.

Table 3.16

Effective collision strengths for H I

T(K)	$1\,^2S, 2\,^2S$	$1\,^2S, 2\,^2P^o$	$1\,^2S, 3\,^2S$	$1\,^2S, 3\,^2P^o$	$1\,^2S, 3\,^2D$
10,000	0.29	0.51	0.066	0.12	0.063
15,000	0.32	0.60	0.071	0.13	0.068
20,000	0.35	0.69	0.077	0.14	0.073

Anderson, H., Balance, C. P., Badnell, N. R., & Summers, H. P. 2000, J.Phys.B, 33, 1255.

3.5 Resulting Thermal Equilibrium

Including all discussed above, the resulting thermal equilibrium is

$$G = L_R + L_{\mathrm{FF}} + L_C \tag{65}$$

- In $n_e \to 0$, $G, L_R, L_{\mathrm{FF}}, L_C$ are all proportional to n_e and to some ion densities n_{ion}.
  So, we can denote $(G, L_R, L_{\mathrm{FF}}, L_C) = n_e n_{\mathrm{ion}}(g, l_R, l_{\mathrm{FF}}, l_C)$, where $(g, l_R, l_{\mathrm{FF}}, l_C)$ don't depend on n_e or n_{ion}.
  Then, we get

  $$g = l_R + l_{\mathrm{FF}} + l_{\mathrm{C}} \tag{66}$$

  Therefore, resulting T does not depend on n_e or n_{ion}.
- In $n_e > n_c(i)$, collisional deexcitation is not negligible.
  $\to$ the cooling rate is decreased.
  $\to$ the equilibrium temperature is increased.

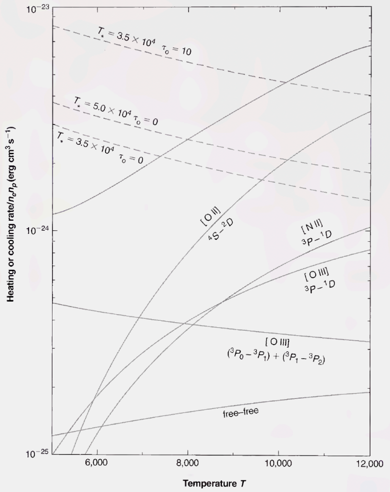
3.5 Example: typical HII region

We set

- $n(\mathrm{O})/n(\mathrm{H}) = 7 \times 10^{-4}, n(\mathrm{Ne})/n(\mathrm{H}) = 9 \times 10^{-5}, n(\mathrm{N})/n(\mathrm{H}) = 9 \times 10^{-5}$
- O, Ne,N : 80% is singly ionized, 20% is doubly ionized.
- $n(\mathrm{H}^0)/n(\mathrm{H}) = 1 \times 10^{-3}$

The resulting cooling rates are shown in figure 3.2

- dependence of T
    - $kT \ll \chi$, the contribution is small.
    - $kT \sim \chi$, it increases rapidly.
    - $kT \gg \chi$, it decreases slowly.
- low T: O^{++} has the greatest contribution.
- high T: O^{+} has the greatest contribution.
- The contribution of H^0 is small in all T.



10⁻²³
10⁻²⁴
10⁻²⁵
$T_* = 3.5 \times 10^4 \quad \tau_0 = 10$
$T_* = 5.0 \times 10^4 \quad \tau_0 = 0$
$T_* = 3.5 \times 10^4 \quad \tau_0 = 0$
Heating or cooling rate/$n_e n_p$ (erg cm^3 s^{-1})
[O II]
$^4S - {}^2D$
[N II]
$^3P - {}^1D$
[O III]
$^3P - {}^1D$
[O III]
$({}^3P_0 - {}^3P_1) + ({}^3P_1 - {}^3P_2)$
free–free
6,000
8,000
10,000
12,000
Temperature T

Figure 3.2

3.5 Example: typical HII region

$$(\text{effective heating rate}) = G - L_R = L_{\text{FF}} + L_C \tag{67}$$

- In figure 3.2, intersection of dashed curve($G - L_R$) and solid curve($L_C + L_{\text{FF}}$) is the solution of equilibrium, and its typical value is $T \sim 7000$K.
- At high n_e, cooling is suppressed by collisional deexcitation and result in T well above that at low n_e.

Appendix: Proof of the lemma

$$\mathbb{E}_{P_2}[X] - \mathbb{E}_{P_1}[X] = \frac{1}{Z_2}\int_{\mathbb{R}} xf(x)g(x)dx - \frac{1}{Z_1}\int_{\mathbb{R}} xf(x)dx \tag{68}$$

$$= \frac{1}{Z_1 Z_2}\left[Z_1 \int_{\mathbb{R}} xf(x)g(x)dx - Z_2 \int_{\mathbb{R}} xf(x)dx\right] \tag{69}$$

$$= \frac{1}{Z_1 Z_2}\left[\int_{\mathbb{R}^2} xf(x)g(x)f(y)dxdy - \int_{\mathbb{R}^2} xf(x)g(y)f(y)dxdy\right] \tag{70}$$

Regarding double integration

$$\int_{\mathrm{R}^2} f(x)f(y)[xg(x) - xg(y)]dxdy =: I \tag{71}$$

we use symmetry trick

$$I = \frac{1}{2}\int_{\mathrm{R}^2} f(x)f(y)[xg(x) - xg(y) + yg(y) - yg(x)]dxdy \tag{72}$$

Appendix: Proof of the lemma

$$I = \frac{1}{2} \int_{\mathrm{R}^2} f(x) f(y)(x - y)(g(x) - g(y)) dx dy \tag{73}$$

If $g(x)$ is increasing,

$$x - y > 0 \quad \Rightarrow \quad g(x) - g(y) > 0 \tag{74}$$

and therefore

$$I > 0 \tag{75}$$

$$\mathbb{E}_{P_2}[X] - \mathbb{E}_{P_1}[X] = \frac{I}{Z_1 Z_2} > 0 \tag{76}$$

$$\mathbb{E}_{P_2}[X] > \mathbb{E}_{P_1}[X] \tag{77}$$

■